ON RADIATIVE HEAT EXCHANGE IN A RANDOMLY INHOMOGENEOUS SCATTERING MEDIUM

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An expression is obtained for the density distribution of the optical thickness in a randomly inhomogeneous medium such as a boiling layer. In the approximation of scattering "forward" and "backward" with respect to the direction of the ray but with allowance for the angular distribution of the radiation, the problem of the transmissivity and reflectivity is solved for the randomly inhomogeneous medium.

A typical example of a randomly inhomogeneous medium is a boiling layer. The transfer of radiation in a randomly inhomogeneous scattering medium in the approximation of weak inhomogeneity was considered in [1, 2].

1. The probability that a photon passes unattenuated a path in a medium with matter density $\rho(x)$ on the ray path is

$$P=e^{-\tau}, \tag{1}$$

where

$$\tau=\beta\int_0^l \rho(x)\,dx \tag{2}$$

is the optical thickness. This probability is determined by averaging the transmission over an ensemble each of whose terms has the same value of $\rho(x)$ along the ray path but for which the distribution of particles in each system of the ensemble is different. The averaging over the ensemble in this case can be replaced by averaging over the area of the pencil of rays. The size of the pencil must be much greater than that of the particles of the medium and the mean distance between them but less than the size of the inhomogeneities of the density $\rho(x)$. In a randomly inhomogeneous medium the optical thickness takes different values τ with probability density $f(\tau)$. The mean value can be found by averaging over an ensemble each of whose members has a different value of $\rho(x)$ along the ray path. Using the ergodicity of the system, the average over the ensemble in the second case can be replaced by averaging with respect to the time. For the mean probability of a photon traversing a path l without attenuation we obtain

$$\langle e^{-\tau}\rangle=\int_0^\infty f(\tau)\,e^{-\tau}d\tau. \tag{3}$$

If the homogeneities are small, then (3) can be represented in the form

$$\langle e^{-\tau}\rangle\simeq e^{-\langle\tau\rangle}\left[1+\frac{\sigma_\tau^2}{2!}\right]. \tag{4}$$

The variance $\sigma_\tau^2=\langle(\tau-\langle\tau\rangle)^2\rangle$ can be expressed in terms of the correlation function of the density of the matter:

$$\sigma_\tau^2=2\beta^2\int_0^l (l-x)\,K_\rho(x)\,dx. \tag{5}$$

The expression (4) with allowance for (5) was used in [1].

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2. To determine the function $f(\tau)$ we use a packet model. Suppose that on the ray path there are n packets with a probability determined in accordance with Poisson's formula, which holds under fairly general assumptions [3]:

$$P_n = \frac{(n_0 l)^n}{n!} e^{-n_0 l}, \tag{6}$$

We approximate the density distribution of the packets with respect to the optical thicknesses for a ray that intersects a packet randomly by the gamma distribution:

$$\varphi(\tau) = \frac{\alpha^{m+1}}{\Gamma(m+1)} \tau^m e^{-\alpha\tau}. \tag{7}$$

The mean optical thickness of a packet is

$$\tau_1 = (m+1)/\alpha. \tag{8}$$

The density distribution with respect to the optical thicknesses for a ray that has passed through the medium has the form

$$f(\tau) = \sum_{n=0}^{\infty} \frac{\langle n \rangle^n e^{-\langle n \rangle}}{n!} f_n(\tau), \tag{9}$$

where

$$f_n(\tau) = \int_0^{\tau} f_{n-1}(\tau - \tau')\varphi(\tau')\,d\tau'; \tag{10}$$

$$f_0(\tau) = \delta(\tau). \tag{11}$$

Performing a Laplace transformation in (9) and noting that

$$f_n^*(s) = [\varphi^*(s)]^n, \tag{12}$$

as a transformation of a convolution [4], we find

$$f^*(s) = e^{-\langle n \rangle} e^{\langle n \rangle \varphi^*(s)}, \tag{13}$$

where

$$\varphi^*(s) = \alpha^{m+1}/(\alpha+s)^{m+1}. \tag{14}$$

Inverting the Laplace transformation, we obtain

$$f(\tau) = e^{-\langle n \rangle}\delta(\tau) + \alpha\langle n \rangle^{1/m+1} e^{-\langle n \rangle - \alpha\tau} \sum_{n=1}^{\infty} \frac{[\alpha\tau\langle n \rangle^{1/m+1}]^{(m+1)n-1}}{n!\Gamma[(m+1)n]}. \tag{15}$$

Using (15), we find the mean value of the optical thickness:

$$\langle \tau \rangle = \langle n \rangle \tau_1, \tag{16}$$

the variance

$$\sigma_\tau^2 = \frac{m+2}{m+1}\frac{\langle \tau \rangle^2}{\langle n \rangle}, \tag{17}$$

and the mean probability of passage of a photon without attenuation:

$$\langle e^{-\tau} \rangle = e^{-\tau_{\text{eff}}}, \tag{18}$$

where

$$\tau_{\text{eff}} = \langle n \rangle \left[1 - \left(\frac{\alpha}{1+\alpha}\right)^{m+1}\right]. \tag{19}$$

The quantity τ_{eff} in a randomly inhomogeneous medium is less than the mean value $\langle\tau\rangle$. The constants τ_1, $\langle n\rangle$, and m must be found experimentally.

If $\langle n\rangle \gg 1$ and $|n - \langle n\rangle| / \langle n\rangle \ll 1$, the distribution (9) goes over into a normal distribution with variance (17). For the proof one notes that in accordance with the central limit theorem the distribution f_n as $n \to \infty$ tends to normal distribution, the Poisson distribution goes over into a Gaussian distribution, and the summation in (9) can be replaced by integration.

3. To calculate radiative heat exchange in a scattering medium we adopt the quasi one-dimensional approximation [7], the essence of which consists of approximating the phase function by the expression

$$\mathscr{P}(\vec{\Omega}\cdot\vec{\Omega}') = a\delta(1+\vec{\Omega}\cdot\vec{\Omega}') + b\delta(1-\vec{\Omega}\cdot\vec{\Omega}'), \tag{20}$$

where a is the probability of backward scattering and b is the probability of forward scattering, i.e., we assume that scattering in an infinitesimal volume of the medium occurs only forward or backward along the ray path. In this approximation the geometry of the medium is taken into account rigorously and the phase function is specified by the single parameter

$$\xi = \frac{b}{a+b}, \tag{21}$$

which determines the fraction of radiation scattered forward in an elementary scattering event.

The solution of the one-dimensional transfer problem is well known [7]. For the coefficient of reflection and transmission it has the form

$$R_1 = R_\infty \frac{1-e^{-2\varkappa\tau}}{1-R_\infty^2 e^{-2\varkappa\tau}}, \tag{22}$$

$$D_1 = \frac{(1-R_\infty^2)e^{-\varkappa\tau}}{1-R_\infty^2 e^{-2\varkappa\tau}}, \tag{23}$$

$$R_\infty = \frac{\varkappa+\gamma-1}{\varkappa-\gamma+1}, \tag{24}$$

$$\varkappa = [1-2\gamma\xi-\gamma^2(1-2\xi)]^{1/2}, \tag{25}$$

where γ is the ratio of the coefficient of scattering to the coefficient of attenuation.

Assuming that hemispherical radiation is incident on the layer, for the transmissivity and reflectivity of the layer in the quasi-one-dimensional approximation we have

$$D = 2(1-R_\infty^2)\int_0^1 \mu \frac{e^{-\varkappa\tau_0/\mu}}{1-R_\infty^2 e^{-2\varkappa\tau_0/\mu}}\, d\mu, \tag{26}$$

$$R = 2R_\infty \int_0^1 \mu \frac{1-e^{-2\varkappa\tau_0/\mu}}{1-R_\infty^2 e^{-2\varkappa\tau_0/\mu}}\, d\mu. \tag{27}$$

Here the medium is assumed homogeneous or with striated-inhomogeneous distribution of the density of the attenuating matter.

In the case $\gamma = 1$ (medium without absorption) we obtain from (26)

$$D(1,\ \tau_0) = 1 - c\tau_0\left[1 - \frac{c\tau_0}{2}\ln\frac{2+c\tau_0}{c\tau_0}\right], \tag{28}$$

where $c = 2(1-\xi)$.

Comparison with the exact solution, obtained in terms of the moments of Ambartsumyan's functions [5], for a spherical phase function shows that the accuracy in the calculation of R and D in the quasi-one-dimensional approximation for a plane layer is better than the accuracy of the Schwarzschild–Schuster approximation except for the case $R(\tau_0 \to \infty) = R_\infty$, when the two approximations are the same.

Using the expansion

$$(1-R_\infty^2 e^{-2\varkappa\tau_0/\mu})^{-1} = 1 + R_\infty^2 e^{-2\varkappa\tau_0/\mu} + \cdots, \tag{29}$$

we can write (25) and (26) in the form

$$D = 2(1+R_\infty^2)\sum_{k=0}^{\infty} R_\infty^{2k} E_3[(2k+1)\varkappa\tau_0], \tag{30}$$

$$R = R_\infty - 2R_\infty(1-R_\infty^2)\sum_{k=0}^{\infty} R_\infty^{2k} E_3[(2k+2)\varkappa\tau_0]. \tag{31}$$

When $\gamma \neq 1$ the series converge rapidly. For example, when $\gamma < 0.8$ it is sufficient to take only two or three terms. Information about the exponential integrals of third order:

$$E_3(x) = \int_0^1 \mu e^{-x/\mu} d\mu \tag{32}$$

can be found, for example, in [6].

4. We assume that on the average the randomly inhomogeneous medium is isotropic. The problem consists of calculating the mean values of R and D for a plane layer. We perform the averaging independently for all rays, using the density (15). It follows from (15) that

$$\langle e^{-g\tau} \rangle = \exp\left\{-\langle n \rangle \left[1 - \left(\frac{\alpha}{\alpha+g}\right)^{m+1}\right]\right\}. \tag{33}$$

This relation is basic for the solution of the problem. Using (32) and (33) and noting that

$$\langle n \rangle = n_0 L/\mu, \tag{34}$$

where L is the thickness of the layer, we find from (30) and (31)

$$\langle D \rangle = 2(1 - R_\infty^2) \sum_{k=0}^{\infty} R_\infty^{2k} E_3 \left\{ n_0 L \left[1 - \left(\frac{\alpha}{\alpha + (2k+1)\varkappa}\right)^{m+1}\right]\right\}, \tag{35}$$

$$\langle R \rangle = R_\infty - 2R_\infty(1 - R_\infty^2) \sum_{k=0}^{\infty} R_\infty^{2k} E_3 \left\{ n_0 L \left[1 - \left(\frac{\alpha}{\alpha + (2k+2)\varkappa}\right)^{m+1}\right]\right\}. \tag{36}$$

At the same time

$$\langle D \rangle \geqslant D(\langle \tau_0 \rangle), \tag{37}$$

$$\langle R \rangle \leqslant R(\langle \tau_0 \rangle). \tag{38}$$

Experiments using a laser beam to probe a boiling layer that I performed in conjunction with R. V. Khomyakov, V. M. Kalemenev, and V. A. Viktorov showed that for a boiling layer one can assume m = 2.

This theory can be used to estimate the influence of fluctuations when one is calculating radiative heat exchange in a rarefied boiling layer. For example, if $n_0 L = 5$, $\alpha = 7$, $m = 2$, $\gamma = 0.5$, and the phase function is spherical, then

$$\langle D \rangle / D(\langle \tau_0 \rangle) = 1.35.$$

The degree of blackness of the layer is

$$\langle \varepsilon \rangle = 1 - \langle D \rangle - \langle R \rangle. \tag{39}$$

NOTATION

$\langle n \rangle$	is the mean number of packets on a ray path in the medium;
n_0	is the mean number of packets on unit length;
l	is the length of a ray path in the medium;
L	is the thickness of the layer;
τ_0	is the optical thickness of the layer;
μ	is the cosine of the angle between the direction of the radiation and the normal to the planes;
τ_1	is the mean optical thickness of a packet;
$f(\tau)$	is the density distribution of the optical thicknesses;
Γ	is the gamma function;
δ	is the delta function;
$\vec{\Omega}$	is a unit vector in the direction of the radiation;
$\langle\ \rangle$	is the mean-value symbol;
τ_{eff}	is the effective optical thickness;
γ	is the ratio of the coefficient of scattering to the coefficient of attenuation;
ξ	is the parameter that characterizes the phase function;
R_∞	is the reflectivity of a semi-infinite medium;
m	is the parameter that characterizes the form of the density distribution;
R, D	are the reflectivity and transmissivity of the layer.

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